Nearly μ-Lindelöf with Respect to a Hereditary Class H

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الملخص

استخدم العديد من علماء الرياضيات استراتيجية الفضاءات التبولوجية المعممة والصفة الوراثية في تمديد البديهيات الأولية للتبولوجيا التقليدية. في هذا العمل، نعرف وندرس مفاهيم بعض من تعميمات فضاءات μ – ليندلوف التبولوجية المعممة بالنسبة لخاصية الوراثة H، تسمى هذه الفضاءات ب فضاءات μ– ليندلوف القريب المعمم بالنسبة لخاصية الوراثةH. بالإضافة إلى تحقيق بعض الخصائص الاساسية للبديهيات الأولية، كذلك سنوضح العلاقة بين هذا العمل وبعض تعميمات μ– ليندلوف الاخرى في الفضاءات التبولوجية المعممة.

Abstract:

The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many mathematicians. In this work, we define and study the concepts of some generalizations of μ-lindelöf generalized topological spaces with respect to a hereditary class H, namely; nearly μ-lindelöf generalized topological spaces with respect to a hereditary class H. In addition, some basic properties of concepts are investigated, the relation among this work and some other generalizations of μ-lindelöf are shown.

Key words: Generalized topological spaces, μ-lindelöfness, a hereditary class H.

1. Introduction and preliminaries

It is well-known that a large number of studies is devoted to the study of classes of subsets of a topological spaces, containing the class of open sets and possessing properties more or less similar to those of open sets. Recently, a significant contribution to the theory of open sets, was introduced by (Császár, 1997). After that, the notions of generalized topology and hereditary classes was studied by ((Császár, 2002, 2005, 2006) and Zahran, El-Saady and Ghareeb, 2012; Kim and Min, 2012, Ramasamy, Rajamani and Inthumathi, 2012)) respectively. Moreover, many authors have been extended the notions of lindelöfness in generalized topological spaces such as; (Sarsak, 2012) (Abuage, and Kiliçman, 2017, 2018). The notions of generalizations of lindelöfnessin term of hereditary classes was studied by (Qahis, 2016). In our work, we define and study some of generalizations of lindelöfness in hereditary generalized topological spaces, namely nearly μ–lindelöf generalized topological spaceswith respect to a hereditary class$\mathcal{H}$.

Definition 1.1. (Császár, 2002) Let X be a nonempty set , *P(X)* denotes the power set of X and μ be a nonempty family of *P(X)*. The symbol μ implies a generalized topology (briefly, *GT)* on X if the empty set $\phi \in \mu$ and $U \in \gamma$ where $\gamma \epsilon \Omega$ implies $\bigcup_{\gamma \epsilon \Omega} U_\gamma \in \mu$. The pair (X, μ) is called generalized topological space (briefly. *GTS*) and we always denote it by *GTS* (X, μ) or X.

Each element of $GT\mu$ is said to be µ-open set and the complement of μ-open set is called μ-closed set. Let A be a subset of a $GTS(X, \mu)$, then $i_\mu(A)$ (resp. $c_\mu(A)$) denotes the union of all μ-open sets contained in A (resp. denotes the intersection of all μ-closed sets containing A), and$X \setminus A$ denotes the complement of A, $c_\mu(X \setminus A) = X \setminus (i_\mu A)$. Moreover, A is said to be μ-regular open (resp. µ-regular closed) if $A = i_\mu c_\mu(A)$ (resp. $A = c_\mu i_\mu(A)$) (Császár, 2008), a subset A of a$GTS(X, \mu)$ is called μ-clopen if it is both μ-open and μ-closed.

Definition 1.2. (Császár, 2004) A $GTS(X, \mu)$ is said to be μ-extremely disconnected if the μ-closure of every μ-open set is μ-

open. If $\beta \subseteq P(X)$ and $\emptyset \in \beta$. Then $\beta$is called a μ-base for μ if $\{\cup \beta' : \beta' \subseteq \beta\} = \mu$, and we say that $\mu$is generated byβ.

Definition 1.3.(Noiri, 2006) Let (X, μ) be a GTS, if a set $X \in \mu$, then a X is called μ-space, and will be denoted by a μ-space (X, μ) or a μ-space X.

Definition 1.4.(Thomas and John, 2012) Let (X, μ) be a *GTS*, a cover $\mathcal{U}$ of a subsets of X is called μ-open cover if the element of $\mathcal{U}$ are μ-open subsets of X .

Definition 1.5. (Kuratowski.,1933) A non-empty family H of subsets of X is called a hereditary class, if $A \in$ H and $B \subset A$ imply that $B \in$H.

Definition 1.6. (Császár, 2006) A $GTS(X, \mu)$ with a hereditary class H, for a subset A of X, the generalized local function of A with respect to H and μ, is defined as follows: $A^* = \{x \in X : U \cap A \notin \mathrm{H}, \ for\ all\ U \in \mu_x\}$, where $\mu_x = \{U : x \in U\ and\ U \in \mu\}$; and the following are defined: $c_\mu^*(A) = A \cup A^*$ and the family $\mu^* = \{A \subset X : X \setminus A = c_\mu^*(X \setminus A)\}$ is a *GT* on X.

The elements of $\mu^*$are called μ^*-open and the complement of a μ^*-open set is called μ^*-closed set. It is clear that a subset A is μ^*-closed if and only if $A^* \subset A$. We call (X, μ, H) a hereditary generalized topological space and denoted by *HGTS*.

Definition 1.7. If(X, μ) be a *GTS* with a hereditary class H, then H is said to be:

(a) (Császár, 2006) μ-co-dense if $\mu \cap \mathrm{H} = \emptyset$,

(b) (Qahis and Noiri, 2017) μ-nowhere dense if $i_\mu c_\mu A = \phi$, for $A \subseteq X$.

We denoted by $\mathrm{H}_C$the hereditary class of countable subsets of X and H_n the hereditary class ofμ- nowhere dense subsets of X.

Definition 1.8. (Sarsak, 2012) Let (X,μ) and $A \subseteq X$. Then a collection $\{U \cap A : U \in \mu\}$ is said to be generalized topology on A, and denoted by μ_A. A $GT\mu_A$ on A forms a generalized topological subspace of X, denoted by(A, μ_A).

Let (X, μ, H) be a H GTS and $A \subseteq X$, $A \neq \phi$. We denoted by $\mathrm{H}_A$the collection

$\{ H \cap (A \cap \Lambda_\mu) : H \in H \}$ and by (A, μ_A, H_A) the subspace of X on A.

Definition 1.9. A $GTS(X,\mu)$ is said to be μ-lindelöf (Sarsak, 2012) (resp. $n\mu$-lindelöfness (Abuage, Kiliçman and Sarsak, 2017)) if for each μ-open cover $\mathcal{U} =\{U_\gamma : \gamma \in \Omega\}$of Λ_μ admits a countable sub-collection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that

$\Lambda_\mu = \bigcup_{n\in\mathbb{N}} U_{\gamma_n}$ (resp. $\Lambda_\mu = \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n})$),

Where Λ_μ is the union of all μ-open sets in X.

A subset A of $GTS(X,\mu)$ is called μ-semi-open if $A \subset c_\mu(i_\mu(A)$ (Császár, 2005).

Definition 1.10. (Qahis, 2016) Let(X,μ) be a GTS and Hbe a hereditary class on X. A H$GTS(X, \mu, H)$is called μH-lindelöf (resp. μH-semi-lindelöf) if each μ-open (resp. μ-semi-open) cover $\{U_\gamma : \gamma \in \Omega\}$ of Λ_μ has a countable subcollection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that $\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}} U_{\gamma_n} \in \mathrm{H}$.

Definition 1.11. (Abuage, and Kiliçman, 2018) Let (X,μ) be a GTS and H be a hereditary class on X. A$HGTS(X, \mu, H)$is called weaklyμ-lindelöf with respect to a hereditary class Hon X ($w\mu H$-lindelöf) if each μ -cover $\{U_\gamma : \gamma \in \Omega\}$of Λ_μ has a countable subcollection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that $\Lambda_\mu \setminus c_\mu(\bigcup_{n\in\mathbb{N}} U_{\gamma_n}) \in \mathrm{H}$.

Theorem 1.1. (Qahis, 2016) Every μH-semi-lindelöf generalized topological space is μH-lindelöf.

Theorem 1.2. (Császár, 2006) Let (X, μ) be a GTS and H be a hereditary class on X

(i) A $GT\mu^*$finer than μ,

(ii) U be a subset of X, if U is μ^*-open, then for each $x \in U^*$ there is $U \in \mu_x$ and $H \in$H such that $x \in U \backslash H \subset U^*$.

2. Nearlyμ-Lindelöf with respect to a hereditary class H

Definition 2.1. Let(X, μ) be a GTS and H be a hereditary class on X. A $HGTS(X, \mu, H)$ is called nearlyμ-lindelöf with respect to a hereditary class H (briefly; $n\mu H$ -lindelöf) if each μ-open cover $\{U_\gamma : \gamma \in \Omega\}$ of Λ_μ has a countable sub-collection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that $\Lambda_\mu \backslash \bigcup_{n\in\mathbb{N}} (i_\mu c_\mu U_{\gamma_n}) \in$H.

Theorem 2.1. A $HGTS(X, \mu, H)$ is $n\mu H$ -lindelöf if and only if every collection $\{F_\gamma : \gamma \in \Omega\}$ of μ-closed sets of X such that $(\bigcap_{\gamma\in\Omega} F_\gamma) \cap \Lambda_\mu = \emptyset$ admits a countable subcollection $\{F_{\gamma_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n\in\mathbb{N}} c_\mu i_\mu F_{\gamma_n}\right) \cap \Lambda_\mu \in$H.

Proof. Necessity, let $\{F_\gamma : \gamma \in \Omega\}$ be a collection of μ-closed sets of X such that$(\bigcap_{\gamma\in\Omega} F_\gamma) \cap \Lambda_\mu = \emptyset$. Then $\Lambda_\mu \subseteq X \backslash (\bigcap_{\gamma\in\Omega} F_\gamma) = \bigcup_{\gamma\in\Omega} (X \backslash F_\gamma)$, i.e., the collection $\{X \backslash F_\gamma : \gamma \in \Omega\}$ is a μ-open cover of Λ_μ. Since $X$is $n\mu H$-lindelöf, there is a countable sub-collection $\{X \backslash F_{\gamma_n} : n \in \mathbb{N}\}$ such that

$$\Lambda_\mu \setminus \left(\bigcup_{n\in\mathbb{N}} (i_\mu c_\mu (X \backslash F_{\gamma_n}))\right) = \Lambda_\mu \setminus \left(\bigcup_{n\in\mathbb{N}} (X \backslash c_\mu i_\mu F_{\gamma_n})\right)$$

$$= \Lambda_\mu \setminus (X \backslash \bigcap_{n\in\mathbb{N}} (c_\mu i_\mu F_{\gamma_n})) \in \text{H}$$

it is obviously to show that:

$\Lambda_\mu \cap (\cap_{n\in\mathbb{N}}(c_\mu i_\mu F_{\gamma_n})) = \Lambda_\mu \setminus (X \setminus \cap_{n\in\mathbb{N}}(c_\mu i_\mu F_{\gamma_n})) \in \mathrm{H}$.

Sufficiency, suppose $\{U_\gamma : \gamma \in \Omega\}$ be a μ-open cover of Λ_μ, then $\Lambda_\mu = \cup_{\gamma\in\Omega} U_\gamma$ and $\{X \setminus U_\gamma : \gamma \in \Omega\}$ is a collection of μ-closed sets of X. Thus $(X \setminus \cup_{\gamma\in\Omega} U_\gamma) \cap \Lambda_\mu = \emptyset$, i.e., $\cap_{\gamma\in\Omega}(X \setminus U_\gamma) \cap \Lambda_\mu = \emptyset$. By hypothesis, there is a countable sub-collection

$\{X \setminus U_{\gamma_n} : n \in \mathbb{N}\}$ such that $(\cap_{n\in\mathbb{N}}(c_\mu i_\mu (X \setminus U_{\gamma_n}))) \cap \Lambda_\mu \in \mathrm{H}$. Since,

$$(\bigcap_{n\in\mathbb{N}}(c_\mu i_\mu (X \setminus U_{\gamma_n}))) \bigcap \Lambda_\mu$$

$$= \Lambda_\mu \setminus (X \setminus (\bigcap_{n\in\mathbb{N}}(c_\mu i_\mu (X \setminus U_{\gamma_n}))))$$

$$= \Lambda_\mu \setminus (X \setminus (X \setminus (\bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}))))$$

$= \Lambda_\mu \setminus \cup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) \in \mathrm{H}$.

Which implies that a $HGTS(X, \mu, H)$ is $n\mu H$-lindelöf.

Theorem 2.2. Let $HGTS(X, \mu, H)$ be a $n\mu H$-lindelöf $HGTS$ and A be a μ-clopen subset of X. Then $(A, \mu_A, H_A)$is $n\mu H_A$-lindelöf.

Proof. Let A be a μ-clopen subset of X, let $\{V_\gamma = U_\gamma \cap A : U_\gamma \in \Omega \; for \; each \; \gamma \in \Omega\}$ be a μ_A-open cover of $A \cap \Lambda_\mu = A$. Hence the family$\{U_\gamma : \gamma \in \Omega\} \cup (X \setminus A)$ forms a μ-open cover of Λ_μ. Since X is $n\mu H$ -lindelöfspace, then there is a countable subfamily $\{U_{\gamma_n} : n \in \mathbb{N}\} \cup (X \setminus A)$such that $\Lambda_\mu \setminus [(\cup_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}) \cup (X \setminus A)] = H \in \mathrm{H}$, now

$$A \cap H = A \cap \left(\Lambda_\mu \setminus \left[(\bigcup_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}) \cup (X \setminus A)\right]\right)$$

$$= A\cap\left(\Lambda_\mu\backslash\left(\bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)\right)\cap\left(\Lambda_\mu \setminus (X \setminus A)\right)$$

$$= A\cap\left(\Lambda_\mu\backslash\left(\bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)\right)\cap\left(\Lambda_\mu\cap A\right)$$

$$= A\cap\left(\Lambda_\mu\backslash\left(\bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)\right)\cap A$$

$$= (A\cap\Lambda_\mu)\cap\left(X \setminus \bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)$$

$$= A\cap\left(X \setminus \bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)$$

$$= A \setminus \left(\bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right) = A \setminus \left(A\cap\left(\bigcup\nolimits_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)\right).$$

However, $\left(\bigcup_{n\in\mathbb{N}} i_\mu c_\mu U_{\gamma_n}\right)\cap A = \bigcup_{n\in\mathbb{N}} i_{\mu_A} c_{\mu_A} V_{\gamma_n}$.Then,

$A\cap H = A\backslash \bigcup_{n\in\mathbb{N}} i_{\mu_A} c_{\mu_A} V_{\gamma_n} \in H_A$, and this proves that a subset A is $n\mu H_A$ -lindelöf.

Theorem 2.3. Let (X,μ) be a μ-space and H be a hereditary class on X. If (X, μ^*, H) *is* $n\mu^* H$-lindelöf then (X, μ, H) is $n\mu H$-lindelöf *HGTS*.

Proof. The proof follows from Theorem 1.1. and Theorem 2.2. (i), since every μ-closed (μ-open) set is μ^*-closed (μ^*-open) set. Thus every μ-clopen set is μ^*-clopen set.

In the following example, we show that the converse of Theorem 2.6. is not true:

Example. Let $\mathbb{R}$ be the all set of real numbers and $\mu = \{U \subset \mathbb{R}: U \text{ is uncountable}\}\cup$

$\{\emptyset\}$ be a GT on $\mathbb{R}$. Suppose $H=\{\mathbb{R}\backslash U: U\in\mu\}$ be a hereditary class on $\mathbb{R}$, observe that $\mathcal{H}$ is not closed under countable union. A μ-space $\mathbb{R}$ is μH -lindelöf (see. (Qahis et al., 2016)) so it is $n\mu H$-lindelöf. For each $t\in\mathbb{R}, \{t\}$ is μ^*–open. Further, $\{t\}$ is μ–closed set so it is μ^*-closed, and hence $c_{\mu^*}i_{\mu^*}(\{t\})=\{t\}$. Further, $\{\{t\}: t\in\mathbb{R}\}$ is a μ^*–open cover of a μ^*–space $\mathbb{R}$. Let that there is a countable collection $\{\{t_n\}: n\in\mathbb{N}\}$ such that $\mathbb{R}\backslash\bigcup_{n\in\mathbb{N}}\{t_n\}\in\mathcal{H}$, and this is not possible. So, a μ^*– space $\mathbb{R}$ is not $n\mu^*H$-lindelöf.

The converse of Theorem 2.3. will be held if a hereditary class $\mathcal{H}$ is closed under countable union as the following:

Theorem 2.4. Let (X,μ) be aμ-space and a hereditary class $\mathcal{H}$ on X is closed under countable union, then (X, μ^*, H) is $n\mu^*H$-lindelöf if and only if (X, μ, H) is $n\mu H$–lindelöf HGTS.

Proof. The necessity is obviously by Theorem 2.3. For sufficiency, suppose (X, μ, H) is $n\mu H$–lindelöf and $\mathcal{H}$ is closed under countable union. Given $\{U_\gamma^*: \gamma\in\Omega\}$ a μ^*–open cover of X, then for each $x\in X, x\in U_{\gamma_x}^*$ for some $\gamma_x\in\Omega$. By Theorem 1.1, (ii) there is $U_{\gamma_x}\in\mu_x$ and $H_{\gamma_x}\in\mathcal{H}$ such that $x\in U_{\gamma_x}\backslash H_{\gamma_x}\subset U_{\gamma_x}^*$. Since the collection $\{U_{\gamma_x}: x\in X\}$ is a μ–open cover of a μH–space X, then there is a countable sub collection $\{U_{\gamma_{x_n}}: n\in\mathbb{N}\}$ such that $X\backslash\bigcup_{n\in\mathbb{N}}(i_\mu c_\mu\, U_{\gamma_{x_n}})=H\in\mathcal{H}$. Since $\mathcal{H}$ is closed under countable union, then $\bigcup\{H_{\gamma_{x_n}}: n\in\mathbb{N}\}\in\mathcal{H}$. Then, $H\cup[\bigcup\{H_{\gamma_{x_n}}: n\in\mathbb{N}\}]\in\mathcal{H}$. Note that

$$X\backslash\bigcup_{n\in\mathbb{N}}(i_{\mu^*}c_{\mu^*}\, U_{\gamma_{x_n}}^*)\subset H\cup[\bigcup\{H_{\gamma_{x_n}}: n\in\mathbb{N}\}]\in\mathcal{H}.$$

So, $X\backslash\bigcup_{n\in\mathbb{N}}(i_{\mu^*}c_{\mu^*}\, U_{\gamma_{x_n}}^*)\in\mathcal{H}$. Thus (X, μ^*, H) is $n\mu^*H$-lindelöf HGTS.

Theorem 2.5. Let (X,μ) be a $GTS$with a hereditary class H, then $(X,\mu)$is $n\mu$-lindelöfif and only if (X,μ,H_c) is $n\mu H_c$-lindelöf $HGTS$.

Proof. The necessity is obviously. Sufficiency, let (X,μ,H_c) is $n\mu H_c$-lindelöf $HGTS$, and $\{U_\gamma : \gamma \in \Omega\}$ be a μ-open cover of Λ_μ. Then by hypothesis, there is a countable subcollection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that $\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) \in$H. Assume,

$\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n})$ pick out U_{γ_i} such that $x_i \in U_{\gamma_i}$ for each $i \in \mathbb{N}$, thus,

$\Lambda_\mu = (\bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n})) \cup (\bigcup_{i\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_i}))$. Which implies that$(X,\mu)$is $n\mu$-lindelöf.

By Theorem above, it is clear that $(X,\mu)$is $n\mu$-lindelöf if and only if $(X,\mu,\{\emptyset\})$ is $n\mu\{\emptyset\}$-lindelöf $HGTS$.

Theorem 2.6. A $HGTS(X,\mu,H)$ is μH-lindelöf then it is $n\mu H$ -lindelöf$HGTS$.

Proof. Let $\{U_\gamma : \gamma \in \Omega\}$ be a μ-open cover of Λ_μ. Since a $HGTS(X,\mu,H)$ is μH-lindelöf then there is a countable sub-collection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that $\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}} U_{\gamma_n} \in$H. But

$$\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}}((i_\mu c_\mu U_{\gamma_n})) \subseteq \Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}}(U_{\gamma_n}).$$

So, $\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}}((i_\mu c_\mu U_{\gamma_n})) \in$H, and the proof is completed.

The converse of above theorem is not true as the following example:

Example. Let $\mathbb{R}$ be the all set of real numbers, $\beta = \{\{a,x\} : x \in \mathbb{R}, a \neq x\}$ and a hereditary classH$= \{\emptyset, \mathbb{R}\}$. If the $GT\mu(\beta)$generated on $\mathbb{R}$ by the μ–base β, then $(\mathbb{R},\mu(\beta),H)$ is $HGTS$, and for each nonempty μ-open set U of $\mathbb{R}$, we have $i_\mu c_\mu U = \mathbb{R}$. So, for each μ –

open cover $\{U_\gamma: \gamma \in \Omega\}$ of $\mathbb{R}$, there is a countable sub-collection $\{U_{\gamma_n}: n \in \mathbb{N}\}$ such that $\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) \in \mathrm{H}$. Thus, $HGTS(\mathbb{R}, \mu(\beta), H)$ is $n\mu H$-lindelöf. Now, $\mathcal{U} = \{\{0, x\}: x \in \mathbb{R}\}$is a μ–open cover of $\mathbb{R}$ and let $\{\{0, x_n\}: n \in \mathbb{N}\}$ be a countable sub-collection of $\mathcal{U}$, it follows that $\mathbb{R} \setminus (\bigcup_{n\in\mathbb{N}}\{0, x_n\}) \notin \mathrm{H}$. Therefore, a $HGTS(\mathbb{R}, \mu(\beta), H)$ is not μH-lindelöf.

Theorem 2.7. Let (X, μ) be a GTS, if

(i) (X, μ) is $n\mu$-lindelöf then $(X, \mu, H_n)$is μH_n-lindelöf.

(ii) (X, μ) is $n\mu$-lindelöf then $(X, \mu, H)$is μH-lindelöf with a μ- co-dense hereditary class H.

Proof. (i) Let (X, μ) is $n\mu$-lindelöf and $\{U_\gamma: \gamma \in \Omega\}$ be a μ-open cover of Λ_μ. Thus there is a countable sub-collection $\{U_{\gamma_n}: n \in \mathbb{N}\}$such that $\Lambda_\mu = \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n})$. Since, $\bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) \subseteq \bigcup_{n\in\mathbb{N}}(c_\mu U_{\gamma_n}) \subseteq c_\mu(\bigcup_{n\in\mathbb{N}} U_{\gamma_n})$, hence,

$\Lambda_\mu \setminus c_\mu(\bigcup_{n\in\mathbb{N}} U_{\gamma_n}) \subseteq \Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) = \emptyset$. So, $i_\mu(\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}} U_{\gamma_n}) = \emptyset$,

and then $\Lambda_\mu \setminus \bigcup_{n\in\mathbb{N}} U_{\gamma_n} \in \mathrm{H}_n$. Which proves that a $HGTS(X, \mu, H_n)$ is μH_n-lindelöf.

(ii) From (i) $\mathrm{H}$is a μ-co-dense hereditary class.

Remark 2.1. By Definitions 1.11 and 2.1, it is clear to show that every $n\mu H$-lindelöf is $w\mu H$-lindelöf but the converse is not true, and we can prove it if the $HGTS$ is restricted to satisfy μ-extremally disconnected and weak P- μ-space as follows:

Definition 2.2. A $\mathrm{GTS}(X, \mu)$ is said to be weak P- μ-space if each countable collection $\{U_{\gamma_n}: n \in \mathbb{N}, \gamma \in \Omega\}$ of μ-open sets in X, then $c_\mu(\bigcup_{n\in\mathbb{N}} U_{\gamma_n}) = \bigcup_{n\in\mathbb{N}}(c_\mu U_{\gamma_n})$.

Theorem 2.8. Let (X, μ) be a μ-extremally disconnected, weak P-μ-space GTS with respect to hereditary class H if X is $w\mu H$-lindelöf then it is $n\mu H$-lindelöf.

Proof. Let $\{U_\gamma : \gamma \in \Omega\}$ be a μ-open cover of Λ_μ , since X is $w\mu H$-lindelöf there is a countable sub-collection $\{U_{\gamma_n} : n \in \mathbb{N}\}$ such that $\Lambda_\mu \backslash c_\mu(\bigcup_{n\in\mathbb{N}} U_{\gamma_n}) \in$H. But X is weak P- μ-space, then $\Lambda_\mu \backslash \bigcup_{n\in\mathbb{N}}(c_\mu U_{\gamma_n}) = \Lambda_\mu \backslash c_\mu(\bigcup_{n\in\mathbb{N}} U_{\gamma_n}) \in$H. Since X is μ-extremally disconnected, thus $\Lambda_\mu \backslash \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) = \Lambda_\mu \backslash \bigcup_{n\in\mathbb{N}}(c_\mu U_{\gamma_n}) \in$H. Then,

$\Lambda_\mu \backslash \bigcup_{n\in\mathbb{N}}(i_\mu c_\mu U_{\gamma_n}) \in$H, which proves that X is $n\mu H$-lindelöf.

Theorem 2.9. Every μH-semi-lindelöf $HGTS$ is $n\mu H$-lindelöf.

Proof. The proof is directly from Theorem 1.1. and Theorem 2.6.

The convers is not true as the following example:

Example. Let $X = \mathbb{R}$, $\mu = \{X, \emptyset, \{0\}\}$ with a hereditary classH$=\{\emptyset, \mathbb{R}\}$. Then (X, μ) is $HGTS$, if for each $x \in X\backslash\{0\}$ let $A_x = \{x, 0\}$ then A_x is μ-semi-open set since $i_\mu A_x = \{0\}$ and $c_\mu i_\mu A_x = X$. Thus the collection $\{A_x : x \in X\backslash\{0\}\}$ is a μ-semi-open cover of X, but has no countable sub-collection $\{A_{x_n} : x \in X\backslash\{0\}\}$, $n \in \mathbb{N}$ such that $\Lambda_\mu \backslash \bigcup_{n\in\mathbb{N}} A_{x_n} \in$H. So, a $HGTS$ $(X, \mu)$is not μH-semi-lindelöf but it is $n\mu H$-lindelöf (even, μH-lindelöf).

Conclusion In this work we defined and studies the notion nearly μ-lindelöf with respect to hereditary class H (briefly, $n\mu H$-lindelöf). Further, some relations among generalizations of μH-lindelöf are studied, since we observed that every $n\mu H$-lindelöf is $w\mu H$-lindelöf, but the opposite is not true. Moreover, ever μH-semi-lindelöf is $n\mu H$-lindelöf and the converse is not true as we showed by example.

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